Pergamon

Stud. Hist. Phil. Sci., Vol. 32, No. 3, pp. 535–555, 2001

Printed in Great Britain
0039-3681/01 $ - see front matter

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Analytical Symbols and Geometrical Figures in Eighteenth-Century Calculus

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Leibnizian–Newtonian calculus was a theory that dealt with geometrical objects; the figure continued to play one of the fundamental roles it had played in Greek geometry: it susbstituted a part of reasoning. During the eighteenth century a process of de-geometrization of calculus took place, which consisted in the rejection of the use of diagrams and in considering calculus as an 'intellectual' system where deduction was merely linguistic and mediated. This was achieved by interpreting variables as universal quantities and introducing the notion of function (in the eighteenth-century meaning of the term), which replaced the study of curves. However, the emancipation of calculus from its basis in geometry was not comprehensive. In fact, the geometrical properties of curves were attributed *de facto* to functions and thus eighteenth-century calculus continued implicitly to use principles borrowed from geometry. There was therefore no transition to a purely syntactical theory based on axiomatically introduced terms, a shift which only took place subsequently in modern times.

Keywords: Eighteenth-century Calculus; Figural Geometry; Analytical Expression; Diagram; Syntax; Semantics.

1. Introduction

Calculus,[1] in its initial phase, was built from a complex mosaic of figures and analytical expressions of curves and geometrical quantities related to them. During the eighteenth century, there was a process of separation between geometry and

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Received 17 August 2000; in revised form 23 November 2000.

[1]Given the objectives of this article, I will use the term 'calculus', without further explanation, to denote that set of techniques, procedures and theories that developed from the earliest research of Newton and Leibniz, which are usually referred to as 'infinitesimal calculus', 'differential and integral calculus', 'infinitesimal calculus', and so on. I also include with this term differential equations and that sector of mathematics that was referred to by Euler as 'the introduction to the analysis of infinites' and subsequently as 'algebraic analysis': the study of functions and their expansions into series without using differentiation and integration. I will continue to use the term 'analysis' when it is present in quotations or when I make specific reference to texts of the period. In any case, the complex historical and philosophical implications of the term 'analysis' are not the subject of this article (on this point, see Otte and Panza, 1997, pp. 365–414).

PII: S0039-3681(01)00009-7

calculus for which the main mathematicians of the period proudly claimed responsibility (see, for example, the quotations from Section 2) and this is amply confirmed by historians.[2] This evolution resulted in the transformation of calculus into an autonomous discipline which seemed to insist on the manipulatory potential of symbolism.

In this article, several aspects of this process will be examined. My aim is to analyse, from the epistemological viewpoint, the different approach between eighteenth-century calculus, mainly in its Eulerian and Lagrangian form, and the earliest forms of calculus of Newton and Leibniz. The causes that produced this transformation are not the subject of this paper; nor are, wherever it is possible to avoid them, the technical details.[3]

My main point is that eighteenth-century calculus was characterised by the conscious elimination of figures. This interpretation of the separation process between calculus and geometry, rather than being reductive, sees the second half of the eighteenth century as marking the definitive maturation of a development which had great importance in the history of mathematics. Indeed, studying the relationship between geometrical objects by referring to diagrams meant making deductions based on the inspection of a specific figure which had a material presence on a sheet of paper. Eighteenth-century calculus, by abandoning the use of figures, became an intellectual system (to use an expression, ‘intellectual’, taken from d’Alembert (1773, vol. 5, p. 154). This means that calculus was understood as a conceptual system where deduction was merely linguistic and mediated, or to put it another way, proceeded from one proposition to another discursively.

My analysis also intends to highlight the elements of continuity between the eighteenth-century concept and the previous one. Indeed, Euler’s and Lagrange’s calculus used notions elaborated from authors such as Viète, Descartes, Leibniz and Newton; nevertheless, figures remained for these authors, despite substantial changes with respect to classical tradition, a crucial moment in the deductive process. After describing the principal notions used to transform calculus into a conceptual system (the variable and the function), I use the final part of the article to examine another essential characteristic of eighteenth-century calculus: although it rejected the use of figures, calculus continued to use principles borrowed from geometry.

In order to clarify the meaning of this statement, I observe that, taking the term ‘geometry’ to refer to the study of objects such as curves, polyhedrons, surfaces and so on, this study can be done, at least in part, by figures: in this case, we could speak of figural geometry, as opposed to non-figural geometry. Non-figural

[2]For instance, Bos stated: ‘From being a tool for the study of curves, analysis developed into a separate branch of mathematics, whose subject matter was no longer the relations between geometrical quantities connected with a curve, but relations quantities in general as expressed by formulas involving letters and numbers’ (Bos, 1974, p. 4).

[3]For these issues, see other published studies, in particular Ferraro (2000).

geometry can use equations, functions, and so on, as an instrument: we would thus have analytical non-figural geometry (also developed in the eighteenth century in relation to the transformations discussed below); the geometrical objects may also be defined starting from appropriate axioms which do not reduce the space to R^3: we would thus have a synthetic non-figural geometry.[4]

Eighteenth-century mathematicians intended by the term ‘geometry’ that which I have referred to above as ‘figural geometry’. Originally, calculus was an integral part of figural geometry: it was considered a special technique of geometry. From 1740, it was regarded as non-figural geometry or ‘symbolic’ geometry (to use another expression of d’Alembert’s[5]): this means that the objects of calculus were no longer curves but functions; however, these were conceived of according to the model of ‘nice’ curves and possessed such properties as continuity, differentiability, and so on which were seen as the analytical translation of properties usually attributed to ‘nice’ curves (the absence of jumps, the existence of tangents, etc.). Moreover, since calculus was regarded as symbolic geometry, it was never conceived of as a formal modern theory, merely syntactic: eighteenth-century mathematicians were guided in their research not by the criterion of syntactical correctness but by the semantic criterion of truth.

2. The Claim of the Independence of Calculus

In the preface to the *Institutiones calculi differentialis*, Euler made two remarkable observations about the nature of differential calculus. First of all, he explicitly rejected geometrical confirmation as a means of testing the validity of calculus, namely, he refused to accept proofs of calculus’ correctness based solely on the fact that calculus reached the same conclusions as elementary geometry: calculus cannot have its own foundation in a geometrical reference (Euler, 1755, p. 6). He then observed:

> I mention nothing of the use of this calculus in the geometry of curved lines: that will be least felt, since this part has been investigated so comprehensively that even the first principles of differential calculus are, so to speak, derived from geometry and, as soon as they had been sufficiently developed, were applied with extreme care to this science. Here, instead, everything is contained within the limits of pure analysis so that *no figure is necessary to explain the rules of this calculus*. (Euler, 1755, p. 9; my emphasis)

Similar statements can be found in Lagrange’s writings. Indeed, in 1773, he wrote: ‘I hope that the solutions I shall give will interest geometers both in terms of the methods and the results. These solutions are purely analytical and can be understood without figures.’(Lagrange, 1773, p. 661) And, in his *Traité de Mécanique analytique*, he stated:

[4]Subtler distinctions (algebraic, differential, descriptive geometry) are not the subject of this article.
[5]See, for example, d’Alembert (1751–80), vol. 7, p. 631.

> One will find no figures in this work. The methods that I present require neither constructions nor geometrical or mechanical reasonings, but only algebraic operations, subject to a regular and uniform course. Those who admire analysis will with pleasure see mechanics become a new branch of it and will be grateful to me for having extended its domain. (Lagrange, 1788, p. 2)

The insistence on figures appears very strange to modern eyes. In modern geometry, figures are dispensable tools for facilitating the comprehension: their role is essentially pedagogical. A modern theory is in fact a conceptual system, composed of explicit axioms and rules of inference, definitions and theorems derived by means of a merely linguistic deduction. Nobody would today consider the absence of figures as a sign that a paper is independent of geometry. In the second half of the eighteenth century, the situation was different. According to d'Alembert, geometry and mechanics were 'material and sensible' science; in particular geometry was 'the science of the properties of extension as it is considered as merely extended and figured' (d'Alembert, 1773, vol. 5; p. 158). Geometry was therefore a science that was studied via sensible figures. On the contrary, the principles of analysis 'were based upon merely intellectual notions, upon ideas that we ourselves shaped by abstraction, by simplifying and generalising the "first" ideas' (d'Alembert, 1773, vol. 5; p. 154).[6]

In the second half of the eighteenth century, calculus was considered as a system of merely intellectual notions, where the term 'intellectual' referred to a form of knowledge that was not based on material awareness but was conceptual and mediated; it functioned in a discursive way along abstract notions. Calculus was distinguished from geometry, which was a line of reasoning applied to figures that were shown in the concrete form of a diagram: figural geometry was entrusted, to a certain extent, to the intuitive immediacy of an inspection of the figure and the perception of the relationships shown in the diagram.

This means of conceiving geometry was, to a degree, classical. In his *The Shaping of Deduction in Greek Mathematics* Netz (1999a) has illustrated in detail how, in Greek geometry, the proof was reasoned upon an object formalised by a figure. Since one reasoned by means of a sensible figure, there was no necessity to clarify precisely all the relationships between the objects of a theory, to make all axioms explicit and to define all terms. The mere inspection of figures provided information that we would now consider missing. Today, for instance, we can state the proposition 'two equal circles of radius r intersect each other if the separation of their centers is less than $2r$' if an appropriate axiom (or a theorem based upon appropriate axioms) guarantees their intersection. Our verbal formulation of geometry implies terms such as circle, radius, centre and so on which only have the meaning attributed to them by the words used to define them and the properties that they derived from their definitions and the axioms of a given theory. Instead, in order

[6]Similarly, Laplace (1821, pp. 131–2) used the term 'intellectual' to refer to the operations of calculus in contrast to the concrete images of geometry.

to derive the existence of the intersection between two circles γ and Γ, Greek geometers could instead refer to the evidence of Figure 1 and simply say: 'Look!' This is precisely what Euclid did in the proof of his very first proposition, where he constructed an equilateral triangle.

As will be seen below, profound changes took place in the seventeenth and eighteenth centuries compared with the Greek world. However, at least part of the geometrical proof continued to be dependent on the figures in the sense that some deductive steps could be inferred by scrutinizing figures, without a verbal formulation.[7]

In claiming[8] the absence of diagrams in their writings, Euler and Lagrange therefore claimed the absence of inference derived from the mere inspection of a figure, and this implies that calculus was conceived of as a exclusively linguistic deductive system. From this viewpoint, eighteenth-century calculus was not simply the linear continuation of Leibniz's or Newton's calculus but was based on a new way of doing mathematics.

3. The Objects of Calculus: Abstract Quantities and Universals

The transformation of calculus into a system based on linguistic deduction was made possible by the notions of variable and function. In the eighteenth century, the terms 'variable' and 'function' had a significantly different connotation with respect to their modern meaning. It is therefore necessary to clarify this meaning which will be discussed in the following paragraph as regards 'variables' and in Section 5 as regards the term 'function'.

In the first works on calculus, a variable[9] was defined as a continually increasing or decreasing quantity. For instance, de l'Hôpital (1696, p. 1) stated: 'Variable quantities are called those which increase or decrease continually whereas constant

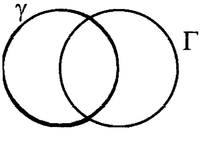

Fig. 1.

[7]I have discussed an interesting example in another paper (Ferraro, 1998, pp. 303–6).

[8]It is well known that, during the Enlightenment, analysis was expressly considered as a well made language (*une langue bien faite*) (see d'Alembert, 1767, vol. 5, p. 164, and Condillac, 1797) and as being superior, in terms of elegance and inventive capacity, to any other language (see Laplace, 1821, p. 132).

[9]In his *Leçon sur le calcul des fonctions*, Lagrange reminded the reader that the dichotomy constant/variable quantity was the result of the evolution of the old Cartesian duality known/unknown quantity: 'In ordinary algebra, one distinguishes quantities into known and unknown. The application of algebra to the theory of curves initially served to distinguish quantities that are present in the equation of a curve as givens, such as axes and parameters, and indeterminate, such as co-ordinates. The same quantities were later considered under the more natural category of constants and variables' (Lagrange, 1806, pp. 11–12).

quantities are those that remain the same while the others change'. Similar definitions lasted during the whole century and Lacroix still wrote in 1797: 'Quantities, considered as changing in value or capable of changing it, are called to be variables, and the name constants is given to those quantities that always maintain their value during the calculation' (Lacroix, 1797, vol. 1; p. 82). This apparent uniformity can be deceptive. Although definitions remained apparently unchanged throughout the century, their meaning was no longer the same because the context in which they were inserted appeared to have altered.

In the seventeenth century, geometrical variable quantities were in fact lines or other geometrical objects connected to a curve, such as the ordinate y, the abscissa x, the arclength s, the subtangent τ, the tangent t, the area A between the curve and the X-axis, and so on. The relations between these variable quantities were embodied in the curve and none of them played a preferential role as an independent variable (Figure 2).[10]

This is also true at the turn of the eighteenth century: thus, in de l'Hôpital's *Analyse des Infiniment petits, pour l'intelligence des lignes courbes*, variables were considered as lines denoted by the letters y, x, . . .

When mathematicians endeavoured to eliminate any reference to geometry, the early concept of a variable become problematic. Moreover, it was not possible to give a numerical meaning to variables because the set of real numbers, as we intend it today, did not yet exist. Only integers and fractions were indeed numbers in the strict sense of the term, while irrational numbers were the ratios of two given quantities of the same kind.[11] In the eighteenth century, real numbers were simply

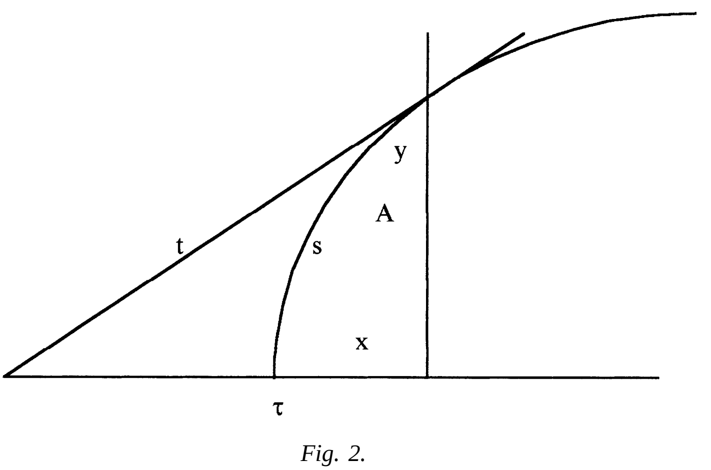

Fig. 2.

[10]On this point, see Bos (1974), pp. 5–6.
[11]See, for example, d'Alembert (1773), vol. 5, p. 188.

tools for denoting and dealing with the quantity which was intended as what can be increased or decreased with continuity.[12]

In order to give to variables a meaning that did not immediately reduce them to lines, eighteenth-century mathematicians resorted to the notion of abstract quantity. The following excerpt from Lagrange helps to clarify the matter:

> When one examines a function with relation to any of the quantities of which it is composed, *one makes the values of this quantity abstract* and considers only the way that it enters in the function, that is to say, how it is combined with itself and the other quantities (Lagrange, 1806, p. 1; my emphasis).

The key word is *abstract*: a variable was an abstract quantity and, therefore, a universal. In the *Introductio in analysin infinitorum*, by using the classic term 'universal', Euler (1748, p. 17) defined a variable quantity as 'an indeterminate or universal quantity, which comprises all determinate values'.

An abstract or universal quantity did not refer to a particular geometric quantity (for example, abscissa or arc length of a given curve) but referred to the quantity in general. It was generated from particular geometrical quantities by means of a process of abstraction, which consists in rendering as a variable what is common to all quantities, just as 'redness' is possessed by all red particular objects. Using an explicitly philosophical language, Euler stated that 'in the same way as the ideas of species and genera are formed from the ideas of individuals, so a variable quantity is the genus, within which all determinate quantities are included' (Euler, 1748, p. 17). According to Aristotle[13] (*Topics* 102a30), a 'genus' is what is predicated in the category of essence of a number of things exhibiting differences in kind; therefore, the notion of a variable concerned the *essence* of quantity and this essence was precisely the capability of being increased or decreased in a continuous way, as the usual eighteenth-century definition of variables stressed.

The fact that a variable was a universal characterised eighteenth-century calculus in a fundamental way. For instance, a variable, being a universal, did not consist simply of the enumeration of its values but substantially differed from them, in the same sense as redness differed from red objects. When a given value was attributed to an abstract quantity, one descended from the general to the particular; the variable lost its essential character of indeterminacy and its nature was altered.

[12]It should be added that, during the eighteenth century, quantities gradually assumed an increasingly strong numerical characterization compared with the seventeenth century, when variable geometric quantities '(and also of physics in that period) were not, or not necessarily, related to a unit and expressed as numbers' (Bos, 1974, p. 5). In the eighteenth century mathematicians were naturally accustomed to working with the decimal representation of real numbers or their approximating sequences and thought that all determinate values of a variable could be expressed as numbers (see, for example, Euler, 1748, vol. 1; pp. 17–18, and vol. 2; p. 289); nevertheless variables were never conceived of as numerical variables.

[13]In this article I will restrict myself to acknowledging the presence of various Aristotelian notions in eighteenth-century calculus without investigating how Euler and Lagrange came to know about them.

This implied that a constant quantity *was not* a *specific case* of a variable quantity: a variable enjoyed its own properties, which might be false for certain determinate values. For instance, in *Théorie des fonctions analytiques* (1813), Lagrange proved that, given a generic function $f(x)$, the development $f(x)+ai+bi^2+ci^3+\ldots+qi^s+\ldots$ of $f(x+i)$ included no fractional power of i. When referring to this theorem, Lagrange asserted: 'This demonstration is general and rigorous as long as x and i remain indeterminate; but this is no longer the case if one gives a determinate value to x' (Lagrange, 1813, p. 23).

What is legitimate for the variable could not be legitimate for all its occasional values. The statement 'x has the property $P(x)$' meant 'x has *naturally* the property $P(x)$'. I use the term 'naturally' in the same sense as was used by Aristotle in *Topics* 134a5ff: 'the man who has said that "biped" is a property of man intends to render the attribute that naturally belongs, . . . "biped" could not be a property of man: for not every man is possessed of two feet.' According to Aristotle, 'what belongs naturally may fail to belong to the thing to which it naturally belongs, as (for example) it belongs to a man to have two feet' (*Topics* 134b5). The existence of men with one foot is not a counterexample for the proposition 'man is biped', since certain men have one foot due to an accident and not by nature (Panza, 1992, pp. 712–713). This approach can be better understood if one considers that a 'property' did not indicate the essence of a thing, which was instead indicated by the definition. The essence of a variable was its capacity to be increased or decreased in a continuous way: if an object x failed to do this, then it was not considered a variable. However, given any property P of x, *there might exist exceptional values at which the property fails*, because P belongs to the variable naturally. Thus, Daniel Bernoulli (1771, p. 84) explained that the sum of a series might exclude any points 'whose existence and location cannot be indicated by abstract analysis. Thus the tangent method cannot indicate cusps if they are in the given curve. As a consequence, however, neither can the tangent method be disproved nor can one be convinced of its falsity'.

Another crucial aspect of variables emerges from Lagrange's quotation cited above. In the eighteenth century, mathematicians considered a variable only 'as it is combined with itself and the other quantities': an abstract quantity was 'merely characterized by its operational relations with other abstract quantities' (Panza, 1996, p. 241). However, two different abstract quantities could differ from each other only by means of symbols x, y, . . . denoting them and not for their specific content which, apart from anything else, was identical for any variable. It is therefore no wonder that the eighteenth-century mathematics usually stressed symbolism, which served to transform the concept of abstract quantities into concrete and manipulable signs (see Section 5, devoted to the notion of function and analytical expression).

The variable as a universal seemed to be the ideal instrument for freeing calculus from the enslavement of figures, giving it the generality that mathematicians of the period had been seeking.[14] Nevertheless, a universal is derived by abstraction from specific and concrete objects: it is different from them but conserves a strong link with them; in no case can it be considered a free creation of our mind. Calculus, since it dealt with universals, was abstract and general but, in the final analysis, always referred to real objects. I will look at the consequences of this in Section 6.

4. Abstract Quantities and Figures in Pre-Eulerian Calculus

The concept of a variable as an abstract quantity was the result of a long evolution which goes back at least to Viète. In his *In artem analyticem isagoge*, Viète distinguished between a *logistice numerosa*, which operates upon numbers, and a *logistice speciosa*, which operated 'with species or forms of things, as, for example, with the letters of the alphabet' (Viète, 1591, p. 328). Jacob Klein has highlighted how the *logistice speciosa* is based on a different conceptualisation of the notion of number compared with that in Greek mathematics. According to Klein, the number in Greek mathematics always meant 'a definite number of definite objects': '*arithmos*' always counted objects (whether they were objects of sense or pure units or monads was of no importance). Viète's *logistice speciosa* operates instead upon an abstract concept of the number[15] and is characterised by the transition from determinacy to indeterminacy. It was a new, general algebra which acted both on continuous 'geometric' magnitudes and numbers divisible into 'discrete units' (Klein, 1968, p. 123).

Although Viète's work marked a fundamental step in the process that led to a conceptualisation of mathematics differing from the Greek one, it still contained aspects that restricted its aspiration towards universality. For example, Viète attached dimension to the species in any given equation and thought that 'only homogeneous magnitudes are to be compared with one another' (Viète, 1591, pp. 324–5). Homogeneity is a form of determination which prevents the reduction of all quantities to only one type of quantity: for example, the square $ABCD$ remained, for Viète, substantially different from the product of the sides AB and CD and cannot be identified with AB^2.

Such a reduction was later used by Descartes by introducing a new definition of multiplication between segments. Taking an arbitrary line segment as the unit segment u, Descartes defined the product of two quantities a and b as the quantity c satisfying the proportion u:a=b:c. This allowed the powers to be interpreted

[14]The rejection of the figural proof was accompanied in the eighteenth century by the gradual rejection of another consolidated demonstrative technique, that of arbitrary exemplification, which consisted in using an appropriate example instead of a general demonstration. This was a typical technique of arithmetic, based on repeatability of the procedure employed which was invariable with respect to the concrete exemplification. According to Netz (1999a, pp. 240–270), Greek geometry (since it used figural demonstration) was also based on repeatability, rather than generality.

[15]For a hypothesis about the causes of this change in relation to Greek mathematics, see Klein (1968), pp. 120–1. A partially different viewpoint is offered by Netz (1999b), pp. 43–45.

appropriately: for example, x^2 was the quantity defined by $1{:}x{=}x{:}x^2$ and any size could be represented by a segment. This led to a kind of dematerialization of the figures themselves[16] which became something substantially different from the 'littered diagrams' of the Greeks. Descartes interpreted 'figures' as structures determined by their symbolic character:[17] the figures were the reification of abstract objects of our thought. Although a figure was conceived of as a symbolic expression, it still consisted in being the icon of a 'determined' geometrical object: this 'imitative' aspect of figures continued to be used by Descartes who effectively attempted to create an integration between signs and figures.[18]

Newton and Leibniz continued to develop Descartes' abstract and symbolic conception, both on the mathematical level in the strict sense as well as on the epistemological level. Nevertheless, there remained a substantial difference from Euler's and Lagrange's conception. To clarify this point, let us consider the following proposition.

(A) Given a curve of equation $y{=}ax^{m/n}$, its area is $\dfrac{na}{m+n}x^{(m+n)/n}$.

Newton (1711, pp. 35–8) justified this proposition on the basis of various assumptions linked to the simple inspection of Figure 3. Indeed, it is the figure that ensures the existence of the area *ADB*, the regular behaviour of the curve (what is referred to today as the continuity of a curve), and the existence of the rectangle *BbKH* whose area is equal to the trapezoid *BbdD*. Certainly, Newton's

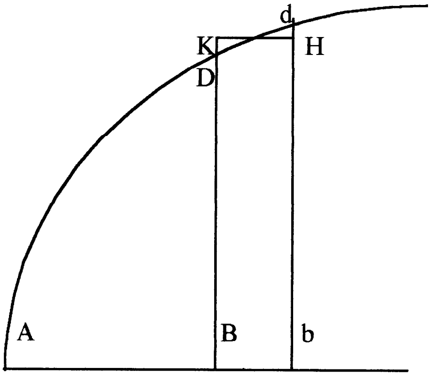

Fig. 3.

[16]Even though quantities were not necessarily expressed as numbers (see note 12), the reduction operated by Descartes was the requirement of the numerability of geometric quantity. Although this process can be considered to have been completed in the seventeenth and eighteenth centuries, we are still a long way from the modern identification of the quantity with *R*.

[17]On this point, see Klein (1968), p. 206.

[18]According to Descartes one could not understand geometrical figures without symbols of algebra, and *vice versa* (Descartes, 1637, pp. 17–18, 20).

demonstration[19] gives considerable attention to the relational aspect and shows a clear algorithmic structure. However, the figure cannot be eliminated although it is only used to represent several properties of an entire class of curves, while the other characteristics are now entrusted to algebraic symbolism.

Overall, Newton's demonstration displays a mixture of figures and analytical symbolism. This mixture is typical of the early years of calculus: it is also found in Leibniz and his school. Nevertheless, Leibniz's epistemological position[20] is worthy of mention because it heralds some of the important developments that definitely influenced mathematicians in the mid-eighteenth century. Leibniz placed great insistence on the fact that calculus could be understood as an algorithm which enabled operations that did not require lines or inspection of figures.[21] But it should be emphasised not only that Leibniz always embedded the algorithm into a geometrical interpretation[22] but that his aim to achieve a calculus without figures was considerably different from the eighteenth-century conception of a self-founding and self-meaning calculus. Leibniz aimed to exclude the inspection of the figure from certain algorithmic procedures concerning geometric objects, but he did not doubt that the algorithm had a meaning and could only be justified to the extent that it referred to such geometric objects.[23] Calculus was an *ars inveniendi*, a method for treating entities that geometry offered (curves and geometric quantities related to them): it was not a theory which had its own objects, in the way that Eulerian calculus had functions.[24] Leibniz's calculus emerged clearly when, for example, he dealt with the problem of the series $1-1+1\ldots$ In a letter to Wolff, he stated, among other things, that the relation $1-1+1\ldots=1/2$ was well founded since a geometric demonstration (*demonstratio linearis*) of it existed.

In conclusion, calculus, in its early phase, developed as an algorithmic and symbolic method based on the notion of general quantity; nevertheless (and this is what

[19]The example discussed here is one of the most analytical to be found in Newton: the role of the figure is, on the one hand, fundamental because it guarantees that the reasoning has a foundation but, on the other hand, only intervenes indirectly in the algorithmic game. The importance of the figure is clearly greater when dealing with calculus with the synthetic method: for this method, my argument is valid *a fortiori*.

[20]If one restricts oneself to mathematical practice, Newton seems, in certain aspects (I refer principally to Newton's *De Methodis Serierum et Fluxionum*, about which see Panza, 1989, pp. 168–170), to anticipate eighteenth-century conceptions even more then Leibniz. If looked at from the epistemological point of view, the situation changes, since Newton gave priority to the synthetic method and this had an effect on the development of his school of thought.

[21]See, for example, Gerhardt (1849–1863), vol. 4, p. 479.

[22]For instance, Guicciardini stated: 'Leibniz always embedded the algorithm into a geometrical intepretation' (Guicciardini, 1999, pp. 167–8). See also Bos (1974), p. 8.

[23]It is well-known that, in Leibniz's opinion, calculus was *cogitatio caeca*: 'blind reasoning'. Operating blindly was conceived of as moving pebbles mechanically in an abacus: an abacus can be used even when the reason why the pebbles are moved in a certain way has been forgotten. Thus, the algorithmic rules of calculus can be used even when their geometrical origin is forgotten. But, just as arithmetic cannot be reduced to the movement of pebbles, calculus cannot be reduced to the manipulation of letters, but these refer to their geometrical meaning.

[24]For an analysis of the importance of the geometrical visualisation of calculus and the difficulties linked to it with regard to Leibniz, see Roero (1995).

distinguishes it from Eulerian and post-Eulerian calculus) it remained a method for studying geometry and did not constitute a self-founding theory: the idea that one could invent a mathematical theory whose aim was the study of quantity in an abstract sense, independent of any figural evidence, did not exist. The objects of Leibniz's and Newton's calculus remained the objects of geometry (analytically expressed), and the figure[25] continued to play one of the fundamental functions of the figure in Greek geometry: a part of the reasoning was unloaded on to it.

5. The Objects of Calculus: Relations and Analytical Expressions

Let us now return to the second half of the eighteenth century. I have already mentioned, at the end of Section 3, that variables, due to their nature as abstract objects, can only be investigated as far as they are placed in relation to themselves and other quantities. Such 'relations' were studied by resorting to the notion of function: calculus therefore assumed the form of a theory of functions. However, as we have already observed, the term had a different meaning in the eighteenth century compared with the modern one.

By 'function' Leibniz initially denoted a line which performs a special duty in a given figure (Youschkevitch, 1976, p. 56). Later, Leibniz used this term to denote a part of a straight line which is cut off by straight lines drawn solely by means of a fixed point and points of a given curve.[26] Therefore, functions were merely geometric variables. Calculus, however, expressed geometric quantities analytically (by 'indeterminates and constants') and, already during the first decades of calculus, mathematicians felt the need to give a name to such analytical expressions of geometric quantities. Thus, while investigating the isoperimetric problem that consists in minimising the area enclosed by a curve, Johann Bernoulli termed them 'functions', with Leibniz's agreement (Gerhardt, 1849–1863, vol. 3, pp. 506–507 and 526). In 1718, he gave the following definition: 'I call a function of a variable quantity, a quantity composed in whatever way of that variable quantity and constants' (Bernoulli, 1718, p. 241). That is, what was termed function was the analytical representation of a geometrical object.

As we have already seen for variables, problems arose if one did not wish to do geometry using analytical methods but tried to establish a foundation for calculus independent from geometry. Thus Euler's definition ('A function of a variable quantity is an analytical expression composed in whatever way of that variable and numbers or constant quantities'—Euler, 1748, vol. 1, section 4) is only apparently similar to Bernoulli's. Indeed, from the 1740s, an analytical expression was no longer the representation of a curve (that embodied the relations between geometrical quantities): it was a representation of a relation between abstract quantities.

[25]This is true even if the figure sometimes had clearly symbolic forms, as in Leibniz's characteristic triangle.

[26]For instance, see Gerhardt (1849–1863), vol. 5, pp. 268, 316.

Now mathematicians could now deal with 'relations' in the sense of purely mental objects, no longer incorporated (at least partially) in material objects such as curves. Analytical expressions were technical instruments that enabled the relations between abstract quantities to be operated upon mathematically.

The fact that many definitions of functions given in the eighteenth century insisted on the technical instrument, the analytical expression, may seem striking. This is the case for the Eulerian definition quoted above and the following one offered by Lagrange: 'The term function of one or more quantities shall be given to every *expression* of calculus to which these quantities belong, with or without other quantities which are considered as given and invariable, so that the quantities of the function can have all possible values' (Lagrange, 1813, p. 15; my emphasis).

However, alongside these definitions, we sometimes encounter, in works by the same author, others with a different tone, even a few line apart, which, in contrast, stress the idea of relation. For instance, in his *Théorie des fonctions analytiques*, Lagrange first stated the above definition; but he was later to assert: 'In general, by the characteristic f or F placed before a variable, we shall denote any function of this variable, that is to say, any quantity *dependent* on this variable and that vary according to it following a given law' (Lagrange, 1813, p. 21; my emphasis).

This diversity of definition has an important basis. The term 'function' as used in the eighteenth century does not indicate just any kind of functional relation but only those relations which can be translated into analytical expressions: only these were studied in calculus. For instance, Lagrange stated:

> The functions x^m, a^x, $\log_a x$, $\sin x$, and $\cos x$ can be regarded as the simple analytical functions of a single variable. All other functions of x are composed of these functions by means of addition, subtraction, multiplication, or division, or are given generally by means of equations in which there are functions of the same form. (Lagrange, 1813, pp. 25–26)

The same concept is expressed in Lagrange (1806) p. 479: a function[27] is not just any kind of relation, but is a relationship that can be expressed through elementary operations.[28]

During the first years of calculus, the set of analytical expressions was even more limited: it consisted of expressions made by composing the variables only by means of algebraical operations. Indeed, until just before the end of the seventeenth century there no formulas for transcendental relation, and these were expressed by means of certain circumlocutions in prose, which basically expressed a method of geometric construction for the curve representing the transcendental

[27]According to Fraser, 'In textbooks of Euler and Lagrange a function is given by a single analytical expression, a formula constructed from variables in finitely many steps using algebraic and transcendental operations and composition of functions,' (Fraser, 1989, p. 325). See also Panza (1996), pp. 250–1.

[28]By 'elementary operations' I mean the six algebraical operations (addition, subtraction, multiplication, division, raising to a power, extraction of root) and some transcendental operations (logarithmic, exponential and trigonometric operations).

relation in question (Bos, 1974, p. 5). The introduction of expressions that contained transcendent operations took place slowly. The mathematicians of the period gave to Johan Bernoulli the credit for having developed calculus of exponential functions and to Euler that for calculus of trigonometric functions.[29] With Euler the set of accepted analytical operations became definitively fixed and coincided with those of elementary operations.

During the eighteenth century, there were attempts to introduce other functions but the results were unsatisfactory. Indeed it was not sufficient to associate a symbol with a 'relation' to obtain automatically an analytical expression and, consequently, a function. A function had to be a known object, in the sense that it was necessary:

(a) to know the manipulatory rules that made it possible to operate with the function;
(b) to be able to calculate the value of the function with sufficient approximation (for example, by means of a specially designed table).[30]

Only if the 'relation' was known to the extent that it could be exhibited as the solution to a problem could the symbolism associated with it be considered an analytical expression and the relation-symbolism complex constitute a function. In practice, no relation, apart from an elementary function, was considered as being known to the extent that it could be thought of as a new function.

At this point, it is important to make clear that the abandonment of the curve, intended in the figural sense, as a fundamental object of calculus was also accompanied by a redefinition of the concept of relation. Indeed, the notion of relation is an extremely natural and ancient one; however, it was, until the eighteenth century, a very vague notion. In particular, it lacked the directional character of modern functions. Modern functions are intended as arrows that move from one single object to another single object according to a defined direction. It was only during the middle of the eighteenth century that relations were introduced that had a directional character (I will refer to them as 'functional relations', in order to avoid using the term function, which in the eighteenth century indicated the association of a functional relation with a suitable symbolism). In the first phase of calculus, a relation between x and y was studied insofar as it formed part of a curve and 'was considered as one entity, not a combination of two mutually inverse mappings $x \longrightarrow y(x)$ and $y \longrightarrow x(y)$' (Bos, 1974, p. 6). The crucial instrument of the analytical manipulation was the equation $F(x,y)=0$, which expressed the curve, and not the function $f(x)$.[31] The fact that the functional relation became the fundamental

[29]On this point, see Katz (1987), pp. 311–324.
[30]For further details, see Ferraro (2000), pp. 115–7.
[31]This conception, for example, leads Newton to formulate the problem of integration in its most complex form; that of solving the fluxional equation, rather than the simple form involved in searching for the fluent.

type of relation studied in mathematics seems to be the natural consequence of the function understood as a composition of elementary functions, since these have a clear directional character.[32]

The introduction of the relational function (together with the fact that it must be given explicitly) made it difficult to interpret $F(x,y)=0$ as a function when it was not possible to make it explicit. This difficulty was felt to the extent that Lagrange and Lacroix referred to an evolution in the notion of function that consisted in the possibility of assuming objects that were not explicitly given (but nevertheless capable of being associated with an expressly given analytical form) as a function.[33]

6. Syntactical and Semantical Aspects in Eighteenth-Century Calculus

We have so far seen that eighteenth-century calculus established its independence from geometry, understood as a figural study of curves, and replaced the study of curves by the study of 'functional relations' between quantities. However, it is necessary to make a further clarification: the functions studied in the eighteenth century were always subject to conditions of 'regularity', or rather (to use modern terms), they were conceived of as continuous, differentiable, and so on. Unlike today, these conditions were not explicitly declared, but were assumed as an integral part of the concept of function. Therefore, for example, it was always assumed that a variable increases and diminishes continuously (it flows, in Newtonian terms) and that, for any function $y(x)$, $\Delta y=y(x+\omega)-y(x)$ is infinitesimal when ω is infinitesimal.[34]

This means that in reality only certain functional relations could be the object of calculus and, specifically, those that ensured *a priori* regularity of the function. Given the particular technical instrument employed, (whereby the analytical expressions were constructed by means of some elementary expressions carefully chosen to incorporate such conditions—Ferraro, 2000, p. 124), it was implicitly

[32]The identification of the directional character of functions was accompanied by the creation of the notions of the function of two or more variable quantities and of the partial derivation, notions that were essential instruments of eighteenth-century calculus.

[33]'Ancient analysts generally include under the denomination of functions of a quantity all the powers of a quantity. Subsequently the sense of this word came to be used in applying it to the results of various algebraic operations: thus, each algebraic expression containing in whatever manner, the products, quotients, powers and roots of these quantities was termed function of one or more quantities. Later, new ideas, resulting from the progress of analysis, led to the following definition of the function. Each quantity whose value depends on one or more quantities is called the function of the latter, both whether one knows or ignores the operations required to obtain from those functions the prior . . . For example, the root of an equation of the fifth degree, for which the expression cannot be assigned given the current state of algebra, is nevertheless a function of the coefficients of the equation since its value depends on the values of the coefficients' (Lacroix, 1797, vol. 1, p. 1).

[34]These conditions were supposed to be valid at least over an interval. On the problem of the extension of valid properties in an interval in global properties, which led to the development of eighteenth-century formalism, see Ferraro (2000), pp. 121–3.

assumed, though never demonstrated, that the conditions of regularity transmitted a generic analytical expression from the elementary expressions.

The conditions to which functions were implicitly subject corresponded with the conditions possessed by curves studied in the figural geometry of the time (the existence of the tangent, the radius of a curve and so on). What eighteenth-century mathematicians created was an abstract translation of higher geometry, a sort of differential geometry: in a sense, it changed the way of doing geometry. D'Alembert referred to calculus as a 'symbolic' geometry: in the terminology used in the introduction, one could refer to it as non-figural geometry. This was a fundamental characteristic of but also a limitation to eighteenth-century calculus.

Naturally, the use of the analytical expression had a profound influence upon the way of dealing with geometric concepts. One example is the famous Eulerian definition of continuity of a curve. Eighteenth-century mathematics inherited the classical concept of Aristotelian continuity.[35] The basic idea behind this conception is that an object is continuous when it is unique. However, there is a substantial difference between applying the idea of uniqueness to the figure and applying it to the relation represented analytically which gives rise to the curve. If we take a hyperbola, a curve constituted by two separate branches, and associate the idea of uniqueness to the object curve, in the physicality of its figural representation, the hyperbola presents a discontinuity. Nevertheless, if we view a hyperbola as a curved line that satisfies an analytically expressed relation, what is important is the uniqueness of the equation[36] from which the hyperbola originates, not the fact that its diagram (by now reduced to a simple expression of an idea) is divided into two parts or not.

Continuity was no longer connected to the observation of the curve but became a conceptual fact. It was precisely the non-figural conception of eighteenth-century calculus that made Euler feel it was necessary to give a definition of continuity. Naturally, the application of the old Aristotelian notion of continuity to the new situation highlighted its inadequacy, with the result that the subsequent developments in mathematics led to its abandonment.

Finally, it should be mentioned that, although calculus is non-figural geometry, it was never a formal theory in the modern sense. Today, a mathematical formal theory is constituted from a set of propositions syntactically derived from the axioms of the theory by means of given rules of derivations. Stating that a proposition 'p' of the mathematical theory T is syntactically correct is not the same as saying that it is semantically true. The truth can be predicated of 'p' if and only if we specify what universes of objects constitute the models of the theory T. In this case, we say that 'p' is true if the event p occurs in the model M where T has been interpreted. Given the theory L_1 containing the statement p and the theory L_2

[35]The notion of continuity in Aristotle has been investigated by M. Panza (1989, pp. 39–65).

[36]The uniqueness of the hyperbola could also be intended as the uniqueness of its construction; however, during the eighteenth century, curves were studied insofar as they were expressed by equations.

containing the statement not-*p*, if one asks: 'May L_1 and L_2 be correct simultaneously?', we today answer that L_1 and L_2 can be syntactically correct at the same time and, even, both true provided they are interpreted by two different models.

In the eighteenth century, mathematicians viewed the matter differently. Their aim was not syntactical correctness but the knowledge of nature. Mathematics was indeed a 'science of nature' (see, for example, the preliminary discourse to the *Encyclopédie* of d'Alembert). The analytical objects of calculus did not exist in virtue of implicit or explicit definitions: they were not a creation of our mind. Despite being abstract, calculus was considered as a mirror of reality; its objects were idealisations derived from the physical world and had an intrinsic existence before and independently from their definition. Mathematical propositions were not merely hypothetical but concerned reality, and were true or false accordingly to whether or not they corresponded to the facts. For instance, d'Alembert stated: 'the physicist ignorant of mathematics considers the truths of geometry as if they were grounded upon arbitrary hypotheses and as mere whims (*jeux d'esprit*) that entirely lack any applications' (d'Alembert, 1773, vol. 5, p. 121). This led to a lack of distinction between syntax and semantics: in the study of calculus, one could refer to its semantic content, reducible to the idea of abstract quantity (and, for this reason, calculus did not specify as a preliminary step all its axioms, as in solely syntactical theories).

Far from being a set of uninterpreted signs without meaning, analytical expressions had a very precise interpretation: they were (abstract) quantities and, thus, had all the properties of quantities, and the rules governing their manipulation were derived by analogy from the rules governing the manipulation of geometric quantities. Eighteenth-century mathematicians never conceived of the possibility of constructing systems of symbols that possessed properties which differed from those of quantities: reality being unique, there could exist a *unique* mathematical structure corresponding to it. Of consequence, in the eighteenth century two theories L_1 and L_2, as defined in the previous page, cannot be correct simultaneously.

7. Conclusion

In this paper, I have illustrated how, during the eighteenth century, calculus abandoned the use of figures as a part of the proof and became a theory which, in principle, could be reduced to a solely linguistic deduction: only in this restricted sense was the process of separation of analysis from geometry real. Indeed, in certain aspects, calculus preserved strict links with geometry. Eighteenth-century calculus was substantially a non-figural geometry.[37] Calculus was the science of

[37]Calculus was appreciated for its greater generality (for example, the symbols $f(x)$, $g(x)$ do not refer to a specific function but to the object function in general; while the diagram of a curve has always its own specificity).

abstract quantities, while figured quantities (that is, quantities represented by a geometric figure) were dealt with by geometry.

Calculus achieved its aim of non-figural geometry by basing itself on the notions of variable and function; these notions were completely different from modern ones. In fact, when, in an eighteenth century text, one encounters a proposition of the kind:

(T) Any function $f(x)$ has the property P,

the expression 'any function' is to be interpreted in a very special way. Firstly, a function was continuous, differentiable, and even analytic (in the modern sense of the term), by its own nature. Secondly, even if we apply a more modern form to (T), such as:

(T_m) If the function $f(x)$ is continuous (or differentiable, or analytic, according to the circumstances) over the interval I, then it has the property P, for every $x \in I$,

we are still very far from the eighteenth century concept. 'Any function $f(x)$' meant not that $f(x)$ was a functional relation dependent upon our discretion but that $f(x)$ was one of known functions (effectively, an elementary function or composition of elementary functions). Besides, the theorem (T) was true if it was assumed that x was a variable and not for particular values c of x, that is, 'isolated exceptional values at which the relation fails are not significant' (Fraser, 1989, p. 331). Consequently it is incorrect to undermine eighteenth-century calculus by means of counterexamples derived from assigning a particular value to a variable, for the simple reason that a theorem of the type (T) was a theorem that concerned abstract quantities (variables) and not their values.

The new conception of calculus as an 'intellectual' system was only the beginning of a lengthy evolution that had profound consequences on the whole of mathematics. I will give just three examples.

Firstly, it was not long before not just higher geometry but the entire field of geometry[38] dealt with symbolic procedures, giving rise to modern analytical geometry.

Secondly, during the seventeenth century, 'the curve was not seen as a graph of a function $x \rightarrow y(x)$, but a figure embodying the relation between x and y' (Bos, 1974, p. 6). This means that a curve, even when it was expressed analytically, was an object that was presented geometrically (figurally). However, during the eighteenth century, the curve began to be seen as a graph of a function $x \rightarrow y(x)$,[39] or,

[38]Initially, the simplest objects of geometry such as the straight line or the circumference were excluded from analytical treatment since analysis was a method for solving complex problems and therefore seemed superfluous to the study of simple questions concerning the straight line. Modern analytical geometry implies the assumption of a non-figural viewpoint even in the treatment of the straight line.

[39]An interesting example is provided by Euler (1797), pp. 10–15. This new method of viewing a curve is probably the requirement for the extensive use of graphs in applied sciences: 'Graphs began to appear around 1770 and became common only around 1820 . . . If we include maps and geometrical

in other words, the materialisation, for didactic and illustrative purposes, of a relation that was conceived of in ideal terms.

Thirdly, the very nature of synthetic geometry was destined to change and become non-figural; that is, it too became a merely linguistic deduction in which the figures were reduced to constituting simply pedagogical support.[40]

Freeing mathematics from every figural consideration and reducing it to linguistic deduction is a complex operation because the absence of the figure makes it impossible to use reasoning that could be entrusted to visual inspection. The reference to the figure must be substituted by adequate axioms and punctual deductions, otherwise lacunae in the deductive process become evident. One merely has to think of the intersection of circles, mentioned in paragraph 1 above; as long as one can reason on the basis of an inspection of a figure, there is no need for particular axioms that guarantee the existence of points of intersection; but, whenever the figure is no longer considered an integral part of the deductive process, the lack of an axiom that guarantees the existence of the intersection emerges as a lacuna in mathematical argumentation.

Eighteenth-century mathematicians made considerable use of semantic interpretation of the objects of calculus and did not yet feel the real need to resolve certain lacunae in the deductive process. It appeared completely obvious to them that a function was continuous or differentiable: they do not seem to have even contemplated the idea that calculus could deal with discontinuous and non-differentiable functions in the modern sense. The lacunae in Euler's or Lagrange's presentation were noticed by mathematicians at the start of the nineteenth century, such as Bolzano, who wrote:

> The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical proposition. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry . . . (Bolzano, 1817, p. 157)

Nevertheless, these words show how the conception of calculus as an inde-

and astronomical diagrams, graphs are very old indeed. What was new in the late eighteenth century was a diagram with rectangular coordinates that showed the relationship between two measured quantities' (Hankins, 1999, pp. 53–4). The graph represents an immediate and easy way of expressing a relation which is fundamentally understood and measured conceptually, and only afterwards represented in diagrammatic form.

[40]This transformation developed in the nineteenth century. The study of the relations between the origins of a mathematics based on the conceptual system and the origins of non-Euclidean geometry is of interest: freeing mathematics of the figure, which by its very nature is Euclidean, may have been an essential requirement for the rise of non-Euclidean geometry.

pendent mathematical theory is one the most important legacies of eighteenth-century mathematics, and a starting point for later development.

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